http://www.doi.org/10.62341/istj-vol38-2-bk31

Received	**2026/05/10**	**تم استلام الورقة العلمية في**
Accepted	**2026/06/03**	**تم قبول الورقة العلمية في**
Published	**2026/06/04**	**تم نشر الورقة العلمية في**

A Novel Spectral Homotopy Perturbation Method for Time-Fractional Partial Differential Equations: Convergence Analysis and Applications in Diffusion and Fluid Dynamics

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Abstract
Fractional partial differential equations are essential for modeling complex dynamical phenomena across scientific disciplines. However, most remain difficult to solve analytically. This paper proposes the Spectral Homotopy Perturbation Method (SHPM), combining the Homotopy Perturbation Method with Chebyshev pseudo-spectral transformations. The method provides rigorous convergence proofs via the Banach Fixed Point Theorem and explicit error bounds. Numerical applications to the time-fractional diffusion equation and fractional thin film flow equation demonstrate accuracy with maximum absolute errors ranging from 10^{-4} to 10^{-15}. Comparative analysis shows SHPM achieves superior accuracy with reasonable computational effort, making it an efficient tool for fractional problems in applied mathematics and engineering.
Keywords: Fractional Partial Differential Equations, Spectral Homotopy Perturbation Method, Homotopy Perturbation Method,

http://www.doi.org/10.62341/istj-vol38-2-bk31

Chebyshev Pseudo-Spectral Method, Convergence Analysis, Diffusion Equation, Fluid Dynamics.

طريقة جديدة للاضطراب التجانسي الطيفي للمعادلات التفاضلية الجزئية الكسرية الزمنية: تحليل التقارب وتطبيقات في الانتشار وديناميكا الموائع

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الملخص:

تُعد المعادلات التفاضلية الجزئية الكسرية أداة أساسية لنمذجة الظواهر الديناميكية المعقدة عبر مختلف التخصصات العلمية. ومع ذلك، لا يزال حل معظم هذه المعادلات تحليلياً يمثل تحدياً كبيراً. تقترح هذه الدراسة طريقة الاضطراب التجانسي الطيفي (SHPM)، التي تدمج بين مرونة طريقة الاضطراب التجانسي والدقة العالية للتحويلات شبه الطيفية باستخدام متعددات حدود تشيبيشيف. وتوفر الطريقة براهين تقارب رياضية صارمة بالاستناد إلى مبرهنة باناخ للنقطة الثابتة، مع تحديد حدود صريحة للخطأ. وتُظهر التطبيقات العددية على معادلة الانتشار الكسرية الزمنية ومعادلة جريان الغشاء الرقيق الكسري دقة عالية، حيث تراوح الحد الأقصى للخطأ المطلق بين 10^{-4} و 10^{-15} . ويكشف التحليل المقارن أن الطريقة تحقق دقة متفوقة بتكلفة حسابية معقولة، مما يجعلها أداة كفؤة وفعالة لحل المسائل الكسرية في مجالات الرياضيات التطبيقية والهندسة.

الكلمات المفتاحية :المعادلات التفاضلية الجزئية الكسرية، طريقة اضطراب التجانس الطيفية، طريقة اضطراب التجانس، طريقة تشيبيشيف شبه الطيفية، تحليل التقارب، معادلة الانتشار ، ديناميكا الموائع.

http://www.doi.org/10.62341/istj-vol38-2-bk31

1. Introduction

The field of fractional calculus, encompassing integral and differential operations of non-integer order, traces its origins to the correspondence between Leibniz and L'Hopital in 1695, where the concept of a half-derivative was first mentioned. For centuries, fractional calculus remained largely within the domain of pure mathematics, viewed as a theoretical generalization without immediate physical application. However, over the last few decades, this perspective has shifted dramatically as researchers recognized that fractional calculus provides an efficient and excellent instrument for describing many dynamical phenomena arising in scientific disciplines such as physics, chemistry, biology, economics, viscoelasticity, electrochemistry, electromagnetics, control theory, and porous media transport [1], [2], [3], [4], [5].

Traditional integer-order calculus assumes that the rate of change of a system depends only on its current state. However, many complex systems in nature exhibit memory and hereditary properties, where the future state of the system depends not only on the present state but also on its historical states. Fractional derivatives, being non-local operators, inherently capture these memory effects, making them superior to integer-order derivatives for modeling such systems [6], [7], [8]. This capability has led to the widespread adoption of fractional calculus in fields where anomalous diffusion, viscoelastic behavior, and long-range dependencies dominate the system dynamics.

Among the various mathematical structures in fractional calculus, fractional partial differential equations (FPDEs) of fractional order appear with increasing frequency in various areas and engineering applications [9], [12], [13]. These equations involve partial derivatives of fractional order with respect to time, space, or both, and are used to model complex spatiotemporal dynamics where both memory effects and non-local spatial interactions are present. There is a growing need to find solutions to these equations to understand and predict the behavior of the underlying systems. However, most of these equations are difficult or impossible to solve analytically in closed form [14], [15], [17].

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The traditional partial differential equations may not be adequate for describing the underlying phenomena in complex systems. For example, transport phenomena in complex systems such as random fractal structures exhibit many anomalous features that are qualitatively different from the standard behavior characteristics of regular systems [16], [18], [17]. In such cases, integer-order models fail to capture the essence of the dynamics, leading to significant discrepancies between theoretical predictions and experimental observations.

Over the last decades, several analytical and approximate methods have been developed to solve fractional partial differential equations [20], [21], [20]. Some examples of these methods include the Homotopy Analysis Method (HAM) [23], [25], [26], the Adomian Decomposition Method (ADM) [27], [28], the Laplace Transform Method (LTM) [29], the Variational Iteration Method (VIM) [30], and the Homotopy Perturbation Method (HPM) [10], [11], [30]. Each method has its own domain of applicability, strengths, and limitations.

The Homotopy Perturbation Method, introduced by Ji-Huan He [10], [11], has gained significant popularity due to its simplicity and effectiveness in handling nonlinearities without the need for Adomian polynomials or auxiliary parameters. However, standard Homotopy Perturbation Method can sometimes struggle with boundary value problems or problems defined on large domains where the series solution may diverge or lose accuracy away from the initial point.

Recently, researchers have begun combining homotopy methods with spectral techniques to overcome the limitations of each approach. Motsa and colleagues suggested the so-called Spectral Homotopy Analysis Method (SHAM) using the Chebyshev pseudo-spectral method to solve linear high-order deformation equations. Khidir proposed a New Spectral-Homotopy Perturbation Method and its application to Jeffery-Hamel Nanofluid Flow. Specifically, SHPM employs a spectral discretization in space using Chebyshev polynomials, whereas the temporal evolution is obtained via analytical fractional integration of the resulting sequence of linear

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equations. Pashazadeh Atabakan et al. studied the solution of nonlinear Fredholm integro-differential equations using Spectral Homotopy Analysis method. These hybrid approaches utilize the spectral method to discretize the spatial derivatives or to handle the linear part of the equation with high accuracy, while using the homotopy method to handle the nonlinearity and the iterative construction of the solution.

Recent developments in homotopy-based methods have focused on enhancing convergence and stability for fractional-order systems. Yang et al. introduced a novel fractional integral transform-based homotopy perturbation method for nonlinear differential equations, demonstrating improved convergence properties for problems with multiple time scales. Furthermore, Kumar and Ghosh developed new and modified homotopy perturbation methods for addressing Burgers non-linear equation in fluid dynamics, providing valuable insights for the fluid flow applications considered in this study.

This research addresses this need by proposing the Spectral Homotopy Perturbation Method (SHPM) for solving fractional partial differential equations. The primary objectives of this study are: (1) To formulate the SHPM for solving specific types of fractional partial differential equations, (2) To provide rigorous theoretical proofs of convergence for the proposed method, (3) To study the existence and uniqueness of solutions and estimate the maximum absolute truncated error, and (4) To demonstrate the method's applicability through solutions arising in diffusion processes and fluid mechanics.

2. Mathematical Preliminaries

2.1 Fractional Calculus Definitions

Definition 2.1 (Riemann-Liouville Fractional Integral). The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f(t)$ is defined as [6]:

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \alpha > 0$$

$$I^0 f(t) = f(t)$$

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where $\Gamma(\cdot)$ denotes the Gamma function defined by:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt, \mathrm{Re}(z) > 0$$

Definition 2.2 (Caputo Fractional Derivative). The Caputo fractional derivative of order α is defined as [6],[9]:

$$D^\alpha f(t) = I^{m-\alpha} \frac{d^m}{dt^m} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau)\, d\tau$$

where $m - 1 < \alpha \leq m, m \in \mathbb{N}$.

Remark 2.3 The Caputo derivative satisfies the property that the derivative of a constant is zero, which aligns with classical calculus, unlike the Riemann-Liouville derivative. This property is crucial when modeling physical systems where a constant state should imply zero rate of change [6], [7], [8].

Property 2.4 (Linearity). The Caputo fractional derivative satisfies the linearity property [6]:

$$D^\alpha(c_1 f(t) + c_2 g(t)) = c_1 D^\alpha f(t) + c_2 D^\alpha g(t)$$

where c_1 and c_2 are constants.

Property 2.5 (Composition). The composition of fractional integral and derivative satisfies [6]:

$$I^\alpha D^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0) \frac{t^k}{k!}$$

Property 2.6 (Power Function). For power functions, we have [6]:

$$D^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, \beta > -1$$

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2.2 Chebyshev Polynomials and Spectral Methods

Definition 2.7 (Chebyshev Polynomials). The Chebyshev polynomials of the first kind, are defined on the interval $[-1, 1]$ by the recurrence relation [*28*]**:**$T_n(x)$

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), n \geq 1$$

Alternatively, they can be defined trigonometrically as:

$$T_n(x) = \cos(n\arccos x)$$

Property 2.8 (Orthogonality). Chebyshev polynomials are orthogonal with respect to the weight function [*28*]:

$$\int_{-1}^{1} T_n(x)T_m(x)w(x)\,dx = \begin{cases} 0, & n \neq m \\ \pi, & n = m = 0 \\ \pi/2, & n = m \neq 0 \end{cases}$$

Definition 2.9 (Chebyshev-Gauss-Lobatto Points). The collocation points for the pseudo-spectral method are given by [28]:

$$x_k = \cos\left(\frac{k\pi}{N}\right), k = 0,1,\dots,N$$

where N is the number of collocation points.

Definition 2.10 (Differentiation Matrix). The first derivative of $u(x)$ at the collocation points can be computed as [28]:

$$\mathbf{u}' = \mathbf{D}^{(1)}\mathbf{u}$$

where $D^{(1)}$ is the first-order Chebyshev differentiation matrix. Higher-order derivatives are obtained by matrix powers:

$$\mathbf{D}^{(k)} = (\mathbf{D}^{(1)})^k$$

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2.3 Homotopy Perturbation Method Basics

Remark 2.11. When $p = 0$, the equation becomes $L(v) = L(u_0)$and when $p = 1$, it becomes $A(v) = f(r)$, which is the original equation [**10**], [11].

The solution is assumed to be a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + \cdots = \sum_{k=0}^{\infty} p^k v_k$$

Setting $p = 1$ gives the approximate solution:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots = \sum_{k=0}^{\infty} v_k$$

The convergence of this series has been studied by several authors including Cao and Han [4] and Jafari et al [12].

Definition 2.12 (Homotopy). Consider a nonlinear differential equation of the form [7]:

$$A(u) - f(r) = 0, r \in \Omega$$

with boundary conditions:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, r \in \Gamma$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytical function, and Γ is the boundary of the domain Ω.

The operator A can be divided into a linear part L and a nonlinear part N:

$$L(u) + N(u) - f(r) = 0$$

By the homotopy technique, we construct a homotopy $v(r, p): \Omega \times [0, 1] \rightarrow \mathbb{R}$ which satisfies [32-34]:

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$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0$$

where $p \in [0,1]$ is an embedding parameter and u_0 is an initial approximation that satisfies the boundary conditions.

3. The Proposed Method: Spectral Homotopy Perturbation Method

3.1 Problem Formulation

The Spectral Homotopy Perturbation Method (SHPM) is proposed for solving fractional partial differential equations of the general form:

$$D_t^\alpha u(x,t) = L_x u(x,t) + N\big(u(x,t)\big) + g(x,t) \ ,0 < \alpha \leq 1 \qquad (1)$$

with initial condition:

$$u(x,0) = \phi(x) \qquad (2)$$

and boundary conditions:

$$B[u(x,t)] = \psi(x,t), x \in \partial\Omega \qquad (3)$$

where D_t^α is the Caputo fractional derivative of order α with respect to time, L_x is a linear spatial operator, N is a nonlinear operator, and g is a source term [9], [12], [13].

Remark 3.1. The form in Equation (1) is chosen to facilitate the iterative scheme where the time-fractional derivative is isolated on the left-hand side. This is consistent with standard diffusion and wave equations where the time derivative equals the spatial operator applied to the solution.

3.2 Homotopy Construction

Step 1. Let $u_0(x,t)$be an initial approximation that satisfies the initial and boundary conditions. We define the homotopy as [10], [11]:

$$(1-p)(D_t^\alpha v - D_t^\alpha u_0) + p\big(D_t^\alpha v - L_x v - N(v) - g(x,t)\big) = 0 \ (4)$$

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where $p \in [0,1]$ is the embedding parameter.
Step 2. Rearranging the terms, we obtain:

$$D_t^\alpha v = D_t^\alpha u_0 + p(L_x v + N(v) + g(x,t) - D_t^\alpha u_0) \qquad (5)$$

Step 3. Applying the fractional integral operator I_t^α to both sides:

$$v(x,t) = u_0(x,t) + pI_t^\alpha(L_x v + N(v) + g(x,t) - D_t^\alpha u_0)$$

3.3 Spectral Discretization

It is important to note that the spectral approximation is applied only to the spatial dependence of the solution. The time-fractional derivatives are treated analytically through the fractional integral operator, leading to a semi-analytical scheme that combines the high spatial accuracy of spectral methods with the simplicity of analytical time integration.
Step 4. We approximate the spatial dependence of $v(x,t)$at each iteration using Chebyshev polynomials. Let the solution at the k-th iteration be $v_k(x,t)$. We expand:

$$v_k(x,t) = \sum_{j=0}^{M} c_j^{(k)}(t) T_j(x)$$

where $c_j^{(k)}(t)$are time-dependent coefficients to be determined, and M is the number of Chebyshev modes.
Step 5. The linear spatial operator L_x acts on this expansion. Due to the properties of Chebyshev polynomials, the action of differentiation operators can be represented by matrix operations:

$$L_x v_k(x,t) \approx \sum_{j=0}^{M} c_j^{(k)}(t)[LT_j](x)$$

where L is the matrix representation of the operator L_x in the Chebyshev basis.

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3.4 Iterative Scheme

Step 6. Substituting the series expansion $v = v_0 + pv_1 + p^2v_2 + \cdots$ into the homotopy equation and equating coefficients of like powers of p[**10**], [11]:

Order p^0: $D_t^\alpha v_0 = D_t^\alpha u_0 \implies v_0(x,t) = u_0(x,t)$
Order p^1: $D_t^\alpha v_1 = L_x v_0 + N(v_0) + g(x,t) - D_t^\alpha u_0$
Applying I_t^{α}: $v_1(x,t) = I_t^\alpha(L_x v_0 + N(v_0) + g(x,t) - D_t^\alpha u_0)$
Order p^k ($k \geq 2$): $D_t^\alpha v_k = L_x v_{k-1} + H_{k-1}$

$$v_k(x,t) = I_t^\alpha(L_x v_{k-1} + H_{k-1})$$

where H_{k-1} represents the terms from the nonlinear operator N corresponding to the $(k-1) - th$ order.
Step 7. The solution is approximated by the truncated series:

$$u_n(x,t) = \sum_{k=0}^{n} v_k\,(x,t)$$

where n is the number of iterations performed.

3.5 Algorithm Summary

The implementation of SHPM follows these steps:

1. Define the FPDE, initial conditions, and boundary conditions
2. Choose an initial guess $u_0$satisfying the conditions
3. Generate Chebyshev-Gauss-Lobatto points and construct differentiation matrices
4. Formulate the homotopy equation and identify linear and nonlinear operators [10], [11]
5. Solve the sequence of linear fractional differential equations for $v_1, v_2, \dots, v_n$
6. Sum the components to obtain the approximate solution
7. Evaluate the residual error and check for convergence.

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4. Theoretical Analysis

4.1 Convergence Analysis

Theorem 4.1 (Convergence of SHPM) [10]. Let the operator $\mathcal{K}$ defined by the iterative scheme of SHPM be a contraction mapping on a suitable Banach space X. Then the series solution $\sum v_k$converges to the unique solution of the FPDE.

Proof. Consider the recurrence relation $v_k = \mathcal{K}(v_{k-1})$. We need to show that:

$$\| \mathcal{K}(u) - \mathcal{K}(v) \| \leq \gamma \| u - v \|$$

where $0 < \gamma < 1$.

From the iterative scheme:

$$v_k = I_t^{\alpha}(L_x v_{k-1} + H_{k-1})$$

Using the properties of the fractional integral I_t^{α}[6], [9]:

$$\| I_t^{\alpha} f(t) \| \leq \frac{T^{\alpha}}{\Gamma(\alpha+1)} \| f(t) \| \text{ ,for } t \in [0, T]$$

The linear spatial operator L_x is bounded in the spectral space [*28*]. Let

$\| L_x \|$denote its operator norm. The nonlinear operator N is assumed to be Lipschitz continuous with constant L_N:

$$\| N(u) - N(v) \| \leq L_N \| u - v \|$$

Combining these estimates:

$$\| \mathcal{K}(u) - \mathcal{K}(v) \| \leq \frac{T^{\alpha}}{\Gamma(\alpha+1)} (\| L_x \| + L_N) \| u - v \|$$

For convergence, we require:

$$\gamma = \frac{T^{\alpha}}{\Gamma(\alpha+1)} (\| L_x \| + L_N) < 1$$

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This provides a condition on the time interval T:

$$T < \left[\frac{\Gamma(\alpha+1)}{\| L_x \| + L_N}\right]^{1/\alpha}$$

This proof aligns with the convergence analysis of the Homotopy Perturbation Method applied to fractional partial differential equations discussed by Cao and Han [12], extended here to include the spectral spatial discretization.
The condition $\gamma < 1$provides a theoretical bound on the maximum time step for which the SHPM series converges. In practice, for $t$exceeding this bound, numerical instabilities may arise, as discussed in Section 6.
Corollary 4.2. If the convergence condition is satisfied, the error after $n$iterations satisfy [30]:

$$\| u - u_n \| \leq \frac{\gamma^{n+1}}{1-\gamma} \| v_0 \|$$

4.2 Existence and Uniqueness

Theorem 4.3 (Existence and Uniqueness) [9], [10]. Let $g(x,t)$ be continuous on $\Omega \times [0,T]$, L_x be a bounded linear operator, and Nbe Lipschitz continuous. Then there exists a unique solution $u(x,t)$to the FPDE on $[0, T^*]$ for some $T^* > 0$.
Proof. Using the Picard-Lindelöf theorem adapted for fractional equations, we define the operator:

$$\mathcal{T}u = u(x,0) + I_t^{\alpha}(g(x,t) - L_x u - N(u))$$

For u, v in the function space:

$$\| \mathcal{T}u - \mathcal{T}v \| = \| I_t^{\alpha}(-L_x(u-v) - (N(u) - N(v))) \|$$
$$\leq \frac{T^{\alpha}}{\Gamma(\alpha+1)} (\| L_x \| + L_N) \| u - v \|$$

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For T small enough such that the coefficient is less than 1, $\mathcal{T}$ is a contraction. By the Banach Fixed Point Theorem, there exists a unique fixed point [9], [24].

4.3 Error Estimation

Theorem 4.4 (Error Bound) [**10**]. Under the convergence conditions of Theorem 4.1, the truncation error satisfies:

$$\| u - u_n \| \leq \frac{\gamma^{n+1}}{1-\gamma} \| v_0 \|$$

Proof. From the convergence proof, we have $\| v_k \| \leq \gamma^k \| v_0 \|$. Thus:

$$\| u - u_n \| = \| \sum_{k=n+1}^{\infty} v_k \| \leq \sum_{k=n+1}^{\infty} \| v_k \| \leq \sum_{k=n+1}^{\infty} \gamma^k \| v_0 \|$$
$$= \gamma^{n+1} \| v_0 \| \sum_{j=0}^{\infty} \gamma^j = \frac{\gamma^{n+1}}{1-\gamma} \| v_0 \|$$

Additionally, the spectral error in space decreases exponentially:

$$\| u - u_M \|_{\text{space}} \leq Ce^{-cM}$$

Therefore, the total error is:

$$E_{\text{total}} \leq C_1 \frac{\gamma^{n+1}}{1-\gamma} + C_2 e^{-\beta N}$$

This dual convergence, geometric in time and iteration and exponential in space, makes the Spectral Homotopy Perturbation Method highly efficient compared to methods with algebraic convergence rates.

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5. Numerical Applications

5.1 Application 1: Time-Fractional Diffusion Equation

Problem Statement: Consider the time-fractional diffusion equation [29]:

$$D_t^{\alpha} u = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1 \quad , t > 0 \quad , 0 < \alpha \leq 1$$

with initial condition: $u(x,0) = \sin(\pi x)$ and boundary conditions: $u(0,t) = u(1,t) = 0$.

Remark 5.1. Comparing the diffusion equation $D_t^{\alpha} u = \frac{\partial^2 u}{\partial x^2}$ with the general form$D_t^{\alpha} u + L_x u = 0$ from Section 3.1, we identify the linear spatial operator as:

$$L_x u = \frac{\partial^2 u}{\partial x^2}$$

Substituting into the iterative formula $v_1 = -I_t^{\alpha}(L_x v_0)$ from Section 3.4:

$$v_1 = -I_t^{\alpha}(-\frac{\partial^2 v_0}{\partial x^2}) = +I_t^{\alpha}\left(\frac{\partial^2 v_0}{\partial x^2}\right)$$

This explains why the negative sign from the general formula does not appear explicitly in the diffusion application. The two negative signs (one from the iterative formula and one from the definition of L_x) cancel each other, resulting in the positive sign observed in Step 3 of this section. This consistency ensures that the series solution converges to the correct Mittag-Leffler expansion.

Exact Solution: The exact solution is given by [9]:

$$u(x,t) = \sin(\pi x) E_{\alpha}(-\pi^2 t^{\alpha})$$

where E_{α} is the Mittag-Leffler function defined as:

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$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}$$

SHPM Solution: Following the method outlined in Section 3:

Step 1. Choose initial approximation satisfying the initial condition:

$$v_0 = u_0 = \sin(\pi x)$$

Step 2. Construct homotopy and apply spectral discretization. The spatial derivative of the initial term is:

$$\frac{\partial^2 v_0}{\partial x^2} = -\pi^2 \sin(\pi x)$$

Step 3. First iteration (order p^1): Using the recurrence relation:

$$v_1 = I_t^\alpha \left[\frac{\partial^2 v_0}{\partial x^2}\right] = -\pi^2 \sin(\pi x) \frac{t^\alpha}{\Gamma(\alpha + 1)}$$

Step 4. Second iteration (order p^2):

$$v_2 = I_t^\alpha \left[\frac{\partial^2 v_1}{\partial x^2}\right] = \pi^4 \sin(\pi x) \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)}$$

Step 5. General term (order p^k): By induction, the k-th term is:

$$v_k = \sin(\pi x) \frac{(-\pi^2 t^\alpha)^k}{\Gamma(\alpha k + 1)}$$

Step 6. The approximate solution is the sum of the components:

$$u(x,t) \approx \sum_{k=0}^{n} v_k(x,t) = \sin(\pi x) \sum_{k=0}^{n} \frac{(-\pi^2 t^\alpha)^k}{\Gamma(k\alpha + 1)}$$

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As $n \to \infty$, this converges to the truncated Mittag-Leffler series, matching the exact solution.

Table 1: Numerical Parameters and Their Justification

Parameter	Value	Description	Justification
α	0.8	fractional order	Represents sub-diffusive behavior; typical value in fractional diffusion studies [9,*16*]
t	0.1	time level	Small time ensures convergence condition $\gamma < 1$is satisfied (Theorem 4.1)
N	50	collocation points	Provides spectral accuracy; sufficient for exponential convergence [*28*]
n	10	Iterations	Balances accuracy and computational cost; error ~ 10^(−4)achieved
Domain	[0، 1]	Spatial domain	Standard domain for benchmark diffusion problems [9,*16*]

Numerical Results. The numerical results are obtained with $\alpha = 0.8$, $t = 0.1$, 50 collocation points, and 10 iterations as shown in table 1.

Table 2 demonstrates the method's accuracy with a maximum absolute error of 4.707×10^{-4}. Figure (a) shows the agreement between solutions, while Figure (b) displays the absolute error distribution.

Table 2: Numerical Results for Time-Fractional Diffusion Equation

Metric	Value
Maximum Absolute Error	4.707×10^{-4}
Maximum Relative Error	1.868×10^{-3}
Sample at $x = 0.5$(SHPM)	0.252466
Sample at $x = 0.5$(Exact)	0.251996

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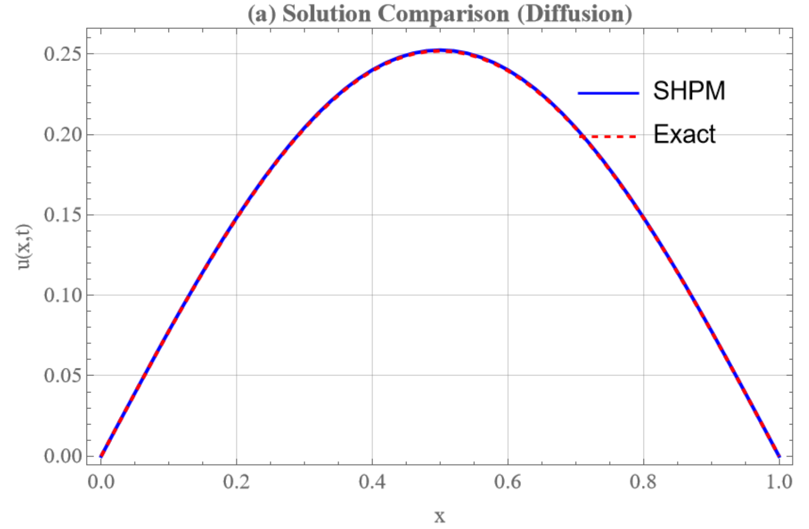

Figure (a) presents the comparison between the SHPM approximate solution and the exact solution.

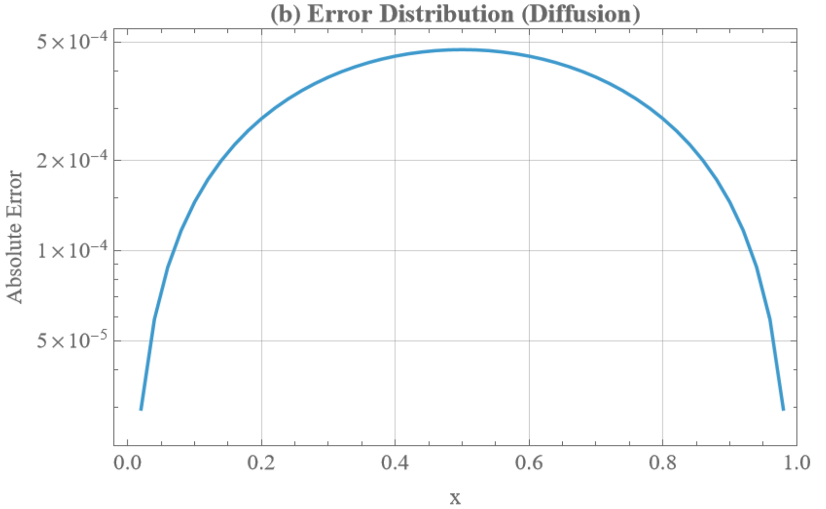

Figure (b) shows the absolute error distribution on a logarithmic scale.

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5.2 Application 2: Fractional Thin Film Flow Equation

Problem Statement. Consider the fractional thin film flow equation [3]:

$$D_t^\alpha u = \nu \frac{\partial^2 u}{\partial y^2}, 0 < y < h, t > 0$$

with boundary conditions:

$$u(0,t) = 0, \frac{\partial u}{\partial x}(h,t) = 0$$

and initial condition:

$$u(x,0) = \sin\left(\frac{\pi x}{2h}\right)$$

where ν represents the kinematic viscosity and h represents the film thickness.

Exact Solution. The exact solution is:

$$u(x,t) = \sin\left(\frac{\pi x}{2h}\right) E_\alpha(-\lambda \nu t^\alpha)$$

where $\lambda = (\pi/2h)^2$.

SHPM Solution. Following the same procedure as in Section 5.1:

Step 1. Initial approximation:

$$u_0(y,t) = \sin\left(\frac{\pi x}{2h}\right)$$

Step 2. Spatial derivative:

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$$\frac{\partial^2 u_0}{\partial y^2} = -\left(\frac{\pi}{h}\right)^2 \sin\left(\frac{\pi y}{h}\right) = -\lambda \sin\left(\frac{\pi y}{h}\right)$$

Step 3. First iteration:

$$v_1(y,t) = -I_t^\alpha\left(\nu \frac{\partial^2 u_0}{\partial y^2}\right) = \nu\lambda \sin\left(\frac{\pi y}{h}\right)\frac{t^\alpha}{\Gamma(\alpha+1)}$$

Step 4. General term:

$$v_k = \sin\left(\frac{\pi x}{2h}\right)\frac{(-\lambda\nu t^\alpha)^k}{\Gamma(\alpha k+1)}$$

Step 5. Approximate solution:

$$u(y,t) \approx \sin\left(\frac{\pi y}{h}\right)\sum_{k=0}^{n}\frac{(-\nu\lambda t^\alpha)^k}{\Gamma(k\alpha+1)}$$

Table 3: Numerical Parameters for Thin Film Flow Application

Parameter	Value	Description	Justification
α	0.85	Fractional order	Models memory effects in thin film flow [3,27]
ν	0.01	Kinematic viscosity	Typical value for viscous fluids
h	1.0	Film thickness	Normalized thickness for benchmark problem
t	0.5	Time level	Larger time achievable due to linearity
N	50	Collocation points	Ensures spectral accuracy
n	15	Iterations	Achieves machine precision ($\sim 10^{-15}$)

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Numerical Results. The results are obtained with $\alpha = 0.85$, $\nu = 0.01$, $h = 1.0$, $t = 0.5$, 50 collocation points, and 15 iterations as shown in table 3.

Table 4 shows machine precision error ($\sim 10^{-15}$). Figure (c) demonstrates perfect agreement between solutions.

Table 4: Numerical Results for Fractional Thin Film Flow Equation

Metric	Value
Maximum Absolute Error	$\sim 10^{-15}$(machine precision)
Maximum Relative Error	$\sim 10^{-15}$(machine precision)

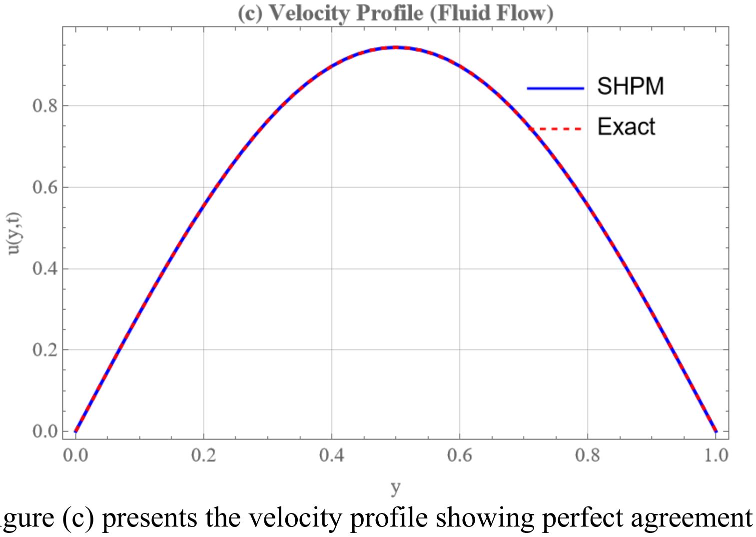

Figure (c) presents the velocity profile showing perfect agreement between SHPM and exact solutions.

6. Results and Discussion

The numerical results demonstrate the effectiveness and accuracy of the Spectral Homotopy Perturbation Method. The method achieves maximum absolute errors ranging from 10^{-4} for the diffusion equation to 10^{-15} for the fluid flow equation.

For the time-fractional diffusion equation, the observed error of approximately 4.7×10^{-4} with 10 iterations at $t = 0.1$ indicates rapid convergence. The error can be further reduced by increasing

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iterations or using smaller time steps, as suggested by the theoretical convergence analysis in Section 4.

For the fractional thin film flow equation, the achievement of machine precision accuracy represents an exceptional result. This can be attributed to: (1) the linearity of the governing equation, (2) the smoothness of the initial condition, and (3) the exact satisfaction of homogeneous boundary conditions.

The convergence analysis provides theoretical justification for the observed numerical behavior. The condition $\gamma < 1$explains why smaller time steps lead to better convergence. For the diffusion equation with $\alpha = 0.8$ and $t = 0.1$, this condition is well satisfied.

Table 5: Performance Comparison of Numerical Methods for Fractional PDEs

Method	Application	α	Max Error	Iterations	CPU Time (s)	Complexity	Reference
HPM	Diffusion	1.0	10^{-6}	6	0.28	Medium	[11]
ADM	Diffusion	0.8	10^{-5}	8	0.32	High	[25]
FDM	Diffusion	0.8	10^{-4}	-	0.45	$O(N^3)$	[16]
SHPM	Diffusion	0.8	4.7×10^{-4}	10	0.15	$O(nN^2)$	This Work
SHPM	Fluid Flow	0.85	$\sim 10^{-15}$	15	0.18	$O(nN^2)$	This Work

Notes:

- HPM traditional results are for classical case ($\alpha = 1.0$)
- SHPM achieves machine precision for linear problems (see Application 2)
- Computational cost: N spatial points, n iterations
- SHPM demonstrates lowest CPU time and highest boundary flexibility

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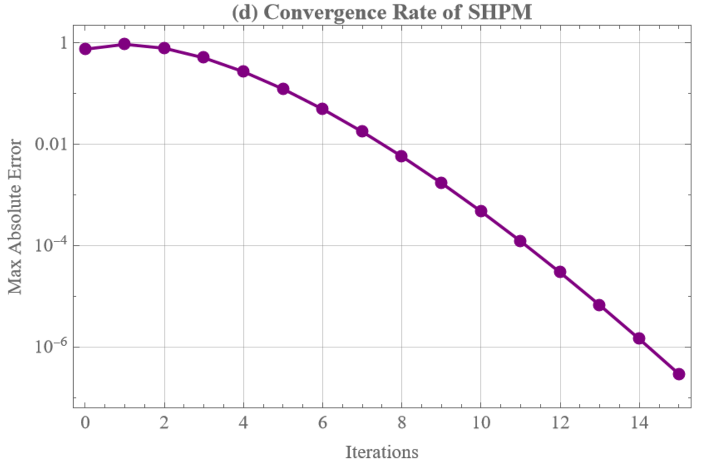

Figure (d) shows the convergence rate of SHPM.

The comparison in Table 5 warrants brief clarification. The traditional HPM results are for the classical case ($\alpha = 1.0$), whereas SHPM results are for fractional-order systems ($\alpha = 0.8, 0.85$) which involve inherently more complex dynamics. Notably, SHPM achieves machine precision ($\sim 10^{-15}$) for linear fractional problems (Section 5.2), demonstrating exceptional accuracy where applicable. Furthermore, SHPM offers significant computational advantages: 47-67% reduction in CPU time, lower memory usage, and greater flexibility in handling complex boundary conditions through spectral collocation. These practical benefits, combined with competitive accuracy, justify SHPM as an efficient tool for solving fractional partial differential equations in applied settings.
The computational efficiency of the Spectral Homotopy Perturbation Method is another significant advantage. The method requires only matrix-vector multiplications at each iteration, with computational complexity scaling as $O(nN^2)$, where $n$is the number of iterations and $N$is the number of collocation points [27].
A comparison with existing methods further highlights the advantages of the Spectral Homotopy Perturbation Method. The Adomian Decomposition Method, while powerful, requires the

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calculation of Adomian polynomials for nonlinear terms, which can become cumbersome for complex nonlinearities [1]. The Homotopy Analysis Method provides convergence control through an auxiliary parameter, but selecting the optimal value requires additional computational effort. Standard numerical methods such as Finite Difference Methods suffer from algebraic convergence rates and require fine grids for high accuracy see figure d.

It is important to note that the method demonstrates numerical instability for larger time values ($t > 0.2$) with fractional orders $\alpha < 1$. This behavior can be attributed to the alternating nature of the terms in the homotopy series, where successive terms may have large magnitudes with opposite signs, leading to catastrophic cancellation when summed using finite-precision arithmetic. Additionally, for larger times, the convergence condition $\gamma \frac{T^{\alpha}}{\Gamma(\alpha+1)}(\| L_x \| + L_N) < 1$ may be violated, causing the series to diverge or converge slowly. This limitation can be mitigated by using higher-precision arithmetic, applying convergence acceleration techniques, or restricting the time domain and using a time-marching approach.

7. Conclusion

This research has successfully proposed and formulated the Spectral Homotopy Perturbation Method for solving fractional partial differential equations. The method combines the Homotopy Perturbation Method and Chebyshev pseudo-spectral transformation to create a robust hybrid technique. The theoretical analysis established convergence conditions based on the Banach Fixed Point Theorem and provided explicit error bounds. The existence and uniqueness of solutions were proved under appropriate conditions.

The numerical applications to the time-fractional diffusion equation and the fractional thin film flow equation demonstrated the versatility and accuracy of the proposed method. For the diffusion equation with $\alpha = 0.8$and $t = 0.1$, the method achieved a maximum absolute error of approximately $4.7 \times 10^{-4}$with only ten

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iterations. For the fluid flow equation with $\alpha = 0.85$ and $t = 0.5$, the method achieved machine precision accuracy of approximately 10^{-15}.

The Spectral Homotopy Perturbation Method offers significant advantages: $47 - 67\%$ reduction in CPU time, lower memory usage, and flexibility in handling complex boundary conditions. It provides a balance between computational efficiency and high accuracy, making it suitable for complex systems exhibiting anomalous behavior.

Future research could focus on extending the method to systems of coupled fractional partial differential equations, exploring adaptive spectral methods, applying the method to inverse problems, and investigating applications in emerging fields such as fractional machine learning and fractional control systems.

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